HIGH LEVEL QUANTILE APPROXIMATIONS OF SUMS OF RISKS

Andrés Cuberos, Esterina Masiello, Véronique Maume-Deschamps

HIGH LEVEL QUANTILE APPROXIMATIONS OF SUMS OF RISKS

A. CUBEROS, E. MASIELLO, AND V. MAUME-DESCHAMPS

Abstract. The approximation of a high level quantile or of the expectation over a high quantile (Value at Risk (VaR) or Tail Value at Risk (TVaR) in risk management) is crucial for the insurance industry. We propose a new method to estimate high level quantiles of sums of risks. It is based on the estimation of the ratio between the VaR (or TVaR) of the sum and the VaR (or TVaR) of the maximum of the risks. We use results on consistently varying functions. We compare the efficiency of our method with classical ones, on several models. Our method gives good results when approximating the VaR or TVaR in high levels on strongly dependent risks where at least one of the risks is heavy tailed.

1. Introduction

Because of regulatory rules (such as Solvency 2 in Europe) or for internal risk management purposes, the estimation of high level quantiles of a sum of risks is of major interest both in finance and insurance industry.
Consider an insurance company that has a portfolio of $d \geq 2$ (possibly) dependent risks which is represented as a random vector $\mathbf{X} = (X_1, \ldots, X_d)$ with cumulative distribution function (c.d.f.) $\mathbf{F}(x_1, \ldots, x_d)$. We assume that all the risks are almost surely positive but we do not assume that they are identically distributed. Let S denote the aggregated risk

$$S = X_1 + \cdots + X_d.$$

We are interested here in the computation of the Value-at-Risk (VaR) and the Tail Value-at-Risk (TVaR) of the sum,

$$\mathrm{VaR}_p(S) = F_S^{\leftarrow}(p) \quad \text{and} \quad \mathrm{TVaR}_p(S) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_u(S)\,\mathrm{d}u,$$

for confidence levels $p \in]0,1[$ near 1, where F_S is the c.d.f. of S and $F^{\leftarrow}$ is its generalized inverse. Problems like this arise for insurance companies, for example, which are required to maintain a minimum capital requirement which is typically calculated as the VaR for the distribution of the sum at some high level of probability. Even when the distribution function $\mathbf{F}$ is known, good estimations for $\mathrm{VaR}_p(S)$ are not trivial since they require a precise calculation of F_S, which is given by the following integral

$$F_S(x) = \int_{\{x_1+\cdots+x_d \leq x\}} \mathrm{d}\mathbf{F}(x_1, \ldots, x_d).$$

Key words and phrases. consistently varying functions, value at risk estimation, risk aggregation.

This integral is more difficult to approximate when d is large and it is usually more efficient to apply Monte Carlo methods to estimate it (for a comprehensive introduction to Monte Carlo methods see [28]). Nevertheless, when p is near 1, the number of replications required to give precise estimations is also large, so new methods are always well received. Classical Extreme Value Theory (EVT) allows to get some estimation of the VaR ([15, 29]). Anyway, using EVT based methods requires an estimation of the EVT parameters, which is known to be not an easy task. Recently, in [7, 17, 8], some approximations on the VaR are obtained for some specific models; see also [18] where theoretical results on the asymptotic behavior of the ratio

$$\frac{\mathrm{VaR}_p(S)}{\sum_{i=1}^{d} \mathrm{VaR}_p(X_i)}$$

are given. Results for the tail distribution of the sum of dependent subexponential risks are obtained in [19] and also in [21] when risks are non-identically distributed and not necessarily positive. In [5], an algorithm to compute the distribution function of S is proposed and in [12], bounds are obtained. Nevertheless, these results may be used to estimate $\mathrm{VaR}_p(S)$ for small dimensions ($d < 4$) and give ranges in dimension 4 or 5. We shall compare our method to the EVT driven ones as well as to the Monte Carlo method, especially for very high level quantiles and in dimension greater than 4 (see Sections 7.2 and 7.3 for simulations in dimension 10 and dimension 150).

Let us denote by M the maximum risk in the portfolio of the company, $M = \max\{X_1, \dots, X_d\}$. The c.d.f. of M, denoted F_M, is given by

$$F_M(x) = \mathbf{F}(x, \dots, x).$$

F_M is directly determined by the c.d.f. $\mathbf{F}$ of the portfolio, so that numerical integration or Monte Carlo methods are not necessary. This also means that $\mathrm{VaR}_p(M)$ can be easily calculated for any given level of confidence p, at most a simple numerical inversion is needed.

In this paper we give some conditions on $\mathbf{X}$ under which the Value-at-Risk and the Tail Value-at-Risk of the sum and maximum are asymptotically equivalent in the sense that there exists some $\Delta \geq 1$ such that

$$\mathrm{VaR}_{1-p}(S) \sim \mathrm{VaR}_{1-\Delta^{-1}p}(M) \quad \text{and} \quad \mathrm{TVaR}_{1-p}(S) \sim \mathrm{TVaR}_{1-\Delta^{-1}p}(M),$$

when $p \to 0$ and where $a(t) \sim b(t)$ when $t \to l$, for $l \in [-\infty, \infty]$ means throughout this paper that $\lim_{t\to l} \frac{a(t)}{b(t)} = 1$. This result is interesting because it allows to estimate the VaR (or TVaR) of the sum by using the VaR (or TVaR) of the maximum, which is easier to calculate, and the estimation of Δ.

For random vectors with common marginals (Fréchet, Gumbel, Weibull) and an Archimedean copula dependence structure [3] and [2] get an asymptotic approximation of the tail of S. These results are generalized in [4] to other dependence structures. In [6], the same results are obtained in the multivariate regularly varying framework. Examples in which the limiting constant Δ can be computed explicitly are also given in [16]. Finally, we would like to mention [23] which is related to our work, in an independent framework and for Pareto marginals.

In this paper, we consider a more general framework with non common marginals and consistently varying tails. We emphasize that our method applies when there are dependences between risks as well as the presence of heavy tailed marginal distributions (see Section 5 for more details). This may be a typical context for risk management applications in insurance and finance. Moreover, the proposed method is tractable, even in high dimension (dimension 150 tested). In Section 7, we compare our method to Monte Carlo one, with the same number of simulations. Of course, by increasing the number of simulations, the approximations would be improved but with the same simulation cost, our method is significantly more efficient than Monte Carlo one. Moreover, we have chosen to use the relation between $\mathrm{VaR}\,(M)$ and $\mathrm{VaR}\,(S)$ to get estimations on $\mathrm{VaR}\,(S)$, using the relation between $\mathrm{VaR}\,(S)$ and $\mathrm{VaR}\,(X_1)$ may seem more natural. In Section 7, we show that our choice is more efficient.

The paper is organized as follows. In Section 2, we recall definitions and classical results on regularly and consistently varying functions. Section 3 contains our main results. In Section 4, we give classes of random vectors satisfying our hypothesis. Section 5 is devoted to a methodology for the estimation of Δ. In Section 6, we give explicit expressions of the VaR on some specific models (introduced in [24, 30] and also considered in [13] where the expression of the VaR is derived). In Section 7, we compare our method with classical ones on several models. Conclusions are given in Section 8. Section A is an appendix which contains useful results on regularly varying functions and their inverses.

2. Preliminaries

In this section, we will first recall the definition of regularly and consistently varying functions and then give some results on consistently varying functions and generalized inverse functions.

Definition 1. Let f be a positive measurable function on $\mathbb{R}_+$.

- We say that f is regularly varying at infinity if there exists a real ρ such that

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^{\rho},$$

for any $t > 0$. This will be denoted by $f \in \mathcal{RV}_\infty(\rho)$. Similarly we say that $f : \mathbb{R}_+ \to \mathbb{R}_+$ is regularly varying at $a \geq 0$ if $f(a + 1/x)$ is regularly varying at infinity. This will be denoted by $f \in \mathcal{RV}_a(\rho)$. If $\rho = 0$ then f is said to be slowly varying.

- We say that f is consistently varying at infinity, denoted as $f \in \mathcal{C}_\infty$, if
$$\lim_{t\downarrow 1} \liminf_{x\to\infty} \frac{f(tx)}{f(x)} = 1.$$
Similarly to the regularly varying definition, f is said to be consistently varying at $a > 0$, denoted by $f \in \mathcal{C}_a$, if $g(x) = f(a + 1/x)$ is consistently varying at infinity.

Examples of regularly varying distributions are Pareto, Cauchy, Burr and stable with exponent $\alpha < 2$. Notice that every regularly varying function is consistently varying. Examples of consistently varying functions which are not regularly varying can be found in [11].

Definition 2. A random variable X with distribution function F is said to have a regularly (consistently) varying upper tail if its survival function $\overline{F}$ is regularly (consistently) varying at infinity.

2.1. **Some results on consistently varying and generalized inverse functions.** It is well known that the sum and composition of regularly varying functions are again regularly varying: if $f_i \in \mathcal{RV}_\infty(\rho_i)$, $i = 1, 2$, then $f_1 + f_2 \in \mathcal{RV}_\infty(\rho)$ with $\rho = \max\{\rho_1, \rho_2\}$ and if $f_2(\infty) = \infty$ then $f_1 \circ f_2 \in \mathcal{RV}_\infty(\rho)$ with $\rho = \rho_1\rho_2$ (see for example [9]). Below we prove that functions which are consistently varying at infinity also satisfy this closure property.

Proposition 2.1. *Let f and g be two non-increasing functions consistently varying at infinity, then the following is satisfied:*

(i) *$g \circ (1/f)$ is consistently varying at infinity if $f(\infty) = 0 = g(\infty)$;*
(ii) *$f + g$ is consistently varying at infinity.*

Proof. (i) Set $\epsilon > 0$ and choose $s' > 1$ and $s > 1$ such that
$$\liminf_{x\to\infty} \frac{g(s'x)}{g(x)} > 1 - \epsilon \text{ and } \liminf_{x\to\infty} \frac{f(sx)}{f(x)} > 1/s'.$$
Then
$$\liminf_{x\to\infty} \frac{g(1/f(sx))}{g(1/f(x))} \geq \liminf_{x\to\infty} \frac{g(s'/f(x))}{g(1/f(x))} > 1 - \epsilon,$$
which proves the proposition.
(ii) Set $\epsilon > 0$, and define for $t > 1$
$$L(t) := \liminf_{x\to\infty} \frac{f(tx) + g(tx)}{f(x) + g(x)} = \liminf_{x\to\infty} \left(\frac{f(tx)/f(x)}{1 + g(x)/f(x)} + \frac{g(tx)/g(x)}{f(x)/g(x) + 1} \right).$$
Then, as f and g are consistently varying at infinity, there exist reals $s > 1$ and $N > 0$ such that $f_i(sx)/f_i(x) > 1 - \epsilon$ for all $x > N$, and $i = 1, 2$. Then
$$L(s) \geq (1 - \epsilon) \liminf_{x\to\infty} \left(\frac{1}{1 + g(x)/f(x)} + \frac{1}{1 + f(x)/g(x)} \right) = (1 - \epsilon).$$
As L is non-increasing, then also $L(s') \geq 1 - \epsilon$ for all $1 < s' < s$. Finally, as $L(t) \leq 1$ for all $t > 1$ we have shown
$$\lim_{t\downarrow 1} L(t) = 1.$$
□

Definition 3. Let f be a non-decreasing function and h a non-increasing function. The generalized inverses of f and h are defined respectively as

$$f^{\leftarrow}(t) = \inf\{s : f(s) \geq t\} \quad \text{and} \quad h^{\leftharpoondown}(t) = \inf\{s : h(s) \leq t\}.$$

Remark that if h is positive we have then $h^{\leftharpoondown}(t) = (1/h)^{\leftarrow}(1/t)$.

Let us recall some well known facts on the generalized inverse functions.

Proposition 2.2. *If f and h are two right-continuous functions, respectively non-decreasing and non-increasing, then for any x and y the following is satisfied:*

(i) $f^{\leftarrow}(f(x)) \leq x$*,* $f(f^{\leftarrow}(x)) \geq x$ *and* $f^{\leftarrow}(y) \leq x \Leftrightarrow y \leq f(x)$
(ii) $h^{\leftharpoondown}(h(x)) \leq x$ *,* $h(h^{\leftharpoondown}(x)) \leq x$ *and* $h^{\leftharpoondown}(y) \leq x \Leftrightarrow y \geq h(x)$.

Proposition 2.3. *If $f_t, t \geq 0$ are non-decreasing functions on $\mathbb{R}$ and $f_t \to f_0$, then $f_t^{\leftarrow} \to f_0^{\leftarrow}$ in the sense that*

$$f_t^{\leftarrow}(x) \to f_0^{\leftarrow}(x)$$

for all x on the continuity points of $f_0^{\leftarrow}$. The same is true for non-increasing functions changing $^{\leftarrow}$ by $^{\leftharpoondown}$ above.

Proof. For the proof on non-decreasing function see for example [25] page 259. Non-increasing case follows using that $f_n^{\leftharpoondown}(x) = (\frac{1}{f_n})^{\leftarrow}(\frac{1}{x})$. □

Below, we prove some results, that are classical for regularly varying functions, for consistently varying functions.

Proposition 2.4. *Let f be a consistently varying and non-increasing measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$ and $a(x)$ and $b(x)$ two positive sequences such that $a(x) \sim b(x)$ as $t \to \infty$ and $a(\infty) = \infty$. Then the following is satisfied:*

(i) *The function $x \mapsto f^{\leftharpoondown}(1/x)$ is consistently varying at infinity*
(ii) *If $f(\infty) = 0$, then $f^{\leftharpoondown} \circ f(x) \sim x$ and $f \circ f^{\leftharpoondown}(1/x) \sim 1/x$ when $x \to \infty$*
(iii) $f(a(x)) \sim f(b(x))$ *and* $f^{\leftharpoondown}(1/a(x)) \sim f^{\leftharpoondown}(1/b(x))$ *when* $x \to \infty$

Proof. (i) Set $H_x(t) = f(tx)/f(x)$ and $H(t) = \liminf_{x\to\infty} H_x(t)$. We will first show that $\liminf_{x\to\infty} H_x^{\leftharpoondown}(\omega) \leq H^{\leftharpoondown}(\omega)$ for all $0 < \omega < 1$. Take $\omega \in (0, 1)$, for each $x > 0$ let us denote by $I_x(\omega)$ the value

$$I_x(\omega) = \inf\{H_s^{\leftharpoondown}(\omega) : s \geq x\}.$$

Then $H_x^{\leftharpoondown}(\omega) \geq I_x(\omega)$ and by the last equivalence of Proposition 2.2 (ii) $\omega \leq H_x(I_x(\omega))$, for all $x > 0$. We have then that $\omega \leq \inf\{H_s(I_s(\omega)) : s \geq x\}$ for any $x > 0$. Notice now that as $I_x(\omega)$ is non-decreasing on x and that each H_x is a non-increasing function then

$$\omega \leq \inf\{H_s(I_s(\omega)) : s \geq x\} \leq \inf\{H_s(I_x(\omega)) : s \geq x\} \leq H(I_x(\omega))$$

for all $x > 0$. By Proposition 2.2 (ii) we have

$$I_x(\omega) \leq H^{\leftharpoondown}(\omega)$$

for all $x > 0$. Thus by taking limits we find

$$\liminf_{x\to\infty} H_x^{\leftharpoondown}(\omega) \leq H^{\leftharpoondown}(\omega).$$

Now, we prove the announced result. For each $x > 0$ and $0 < \omega < 1$ we have $H_x{}^{\leftarrow}(\omega) = f^{\leftarrow}(\omega f(x))/x$. Proposition 2.2 (ii) gives, $f \circ f^{\leftarrow}(1/x) \leq 1/x$ for all $x > 0$ and then for $0 < \omega < 1$ we have $f^{\leftarrow}(\omega/x) \leq f^{\leftarrow}(\omega f \circ f^{\leftarrow}(1/x))$. Now, we have

$$1 \leq \liminf_{x\to\infty} \frac{f^{\leftarrow}(\omega/x)}{f^{\leftarrow}(1/x)} \leq \liminf_{x\to\infty} \frac{f^{\leftarrow}(\omega f \circ f^{\leftarrow}(1/x))}{f^{\leftarrow}(1/x)} = \liminf_{x\to\infty} H_x^{\leftarrow}(\omega) \leq H^{\leftarrow}(\omega).$$

As $f \in \mathcal{C}_\infty$ implies $H(t) \uparrow 1$ when $t \downarrow 1$, and thus, as H is non-increasing, then $H^{\leftarrow}(\omega) \downarrow 1$ when $\omega \uparrow 1$ and

$$\lim_{\omega\uparrow 1} \liminf_{x\to\infty} \frac{f^{\leftarrow}(\omega/x)}{f^{\leftarrow}(1/x)} = 1,$$

which proves the Proposition.
(ii) Proposition 2.2 (ii) gives $f^{\leftarrow}(f(x)) \leq x$ for all $x > 0$. By definition of the generalized inverses it follows that

$$0 \leq x - f^{\leftarrow} \circ f(x) \leq \lim_{s\uparrow 1} f^{\leftarrow}(sf(x)) - \lim_{t\downarrow 1} f^{\leftarrow}(tf(x))$$

where the last expression represents the size of the possible jump of $f^{\leftarrow}$ at $f(x)$. Take $\epsilon > 0$, by (i) the mapping $x \mapsto f^{\leftarrow}(1/x)$ is consistently varying so we can choose $N > 0$, $0 < s < 1$ and $t > 1$ such that for all $x > N$

$$\frac{f^{\leftarrow}(s/x)}{f^{\leftarrow}(1/x)} \leq 1 + \frac{\epsilon}{2} \quad \text{and} \quad \frac{f^{\leftarrow}(t/x)}{f^{\leftarrow}(1/x)} \geq 1 - \frac{\epsilon}{2}.$$

Then

$$0 \leq 1 - \frac{f^{\leftarrow} \circ f(x)}{x} \leq \frac{f^{\leftarrow}(sf(x)) - f^{\leftarrow}(tf(x))}{f^{\leftarrow} \circ f(x)} \leq \epsilon$$

where the last inequality holds for all $x > f^{\leftarrow}(1/N)$. As ϵ was arbitrary it had been proven that $f^{\leftarrow} \circ f(x) \sim x$. The proof that $f \circ f^{\leftarrow}(1/x) \sim 1/x$ is similar.
(iii) Set $\epsilon > 0$, and let T be such that $1 - \epsilon \leq a(x)/b(x) \leq 1 + \epsilon$ for any $x \geq T$. Then as f is non-increasing, for $x \geq T$ we have

$$\frac{f((1+\epsilon)b(x))}{f(b(x))} \leq \frac{f(a(x))}{f(b(x))} \leq \frac{f((1-\epsilon)b(x))}{f(b(x))}.$$

Applying limits in the equation above we get,

$$\liminf_{x\to\infty} \frac{f((1+\epsilon)b(x))}{f(b(x))} \leq \liminf_{x\to\infty} \frac{f(a(x))}{f(b(x))}$$

and

$$\limsup_{x\to\infty} \frac{f(a(x))}{f(b(x))} \leq \limsup_{x\to\infty} \frac{f((1-\epsilon)b(x))}{f(b(x))}.$$

As $f \in \mathcal{C}_\infty$ then

$$\lim_{\epsilon\downarrow 0} \liminf_{x\to\infty} \frac{f((1+\epsilon)b(x))}{f(b(x))} = 1.$$

Similarly, as

$$\limsup_{x\to\infty} \frac{f((1-\epsilon)b(x))}{f(b(x))} = \liminf_{x\to\infty} \frac{f(b(x))}{f((1-\epsilon)b(x))}$$

then $f \in \mathcal{C}_\infty$ implies

$$\lim_{\epsilon\downarrow 0} \limsup_{x\to\infty} \frac{f((1-\epsilon)b(x))}{f(b(x))} = 1.$$

Thus we had proved

$$\lim_{x\to\infty} \frac{f(a(x))}{f(b(x))} = 1.$$

The proof of $f^{\leftarrow}(1/a(x)) \sim f^{\leftarrow}(1/b(x))$ follows from (i). □

3. Asymptotic results on the VaR and the TVaR of the sum and the maximum

Our main results are now stated: they link the Value-at-Risk of the sum and the maximum in case where the survival function of the maximum, $\overline{F}_M$, is consistently varying. The results still hold for the TVaR. Recall that we do not assume that the marginal distributions are either identically distributed or independent.

Theorem 3.1. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a vector of positive random variables (r.v.s). Suppose that $\overline{F}_M$ is consistently varying and that $\delta(x) := \frac{\overline{F}_S(x)}{\overline{F}_M(x)} \to \Delta$ as $x \to \infty$, for some $1 \le \Delta < \infty$. Then*

$$\mathrm{VaR}_{1-p}(S) \sim \mathrm{VaR}_{1-\Delta^{-1}p}(M) \text{ as } p \to 0.$$

The same result applies for the TVaR *if* $\mathrm{TVaR}_p(M)$ *exists for all p. If moreover $\overline{F}_M$ is $\rho-$varying then $\Delta \le d^{-\rho}$.*

Let us mention that in the case of regularly varying functions $\overline{F}_M$, the first part of Theorem 3.1 follows from properties of regularly varying functions and the second part follows from Karamata's Theorem. The insterest of Theorem 3.1 is that it holds for consistently varying functions $\overline{F}_M$.

Proof. By Proposition 2.4 (ii) and (iii) we have

$$t \sim \overline{F}_M^{\leftarrow} \circ \overline{F}_M(t) \sim \overline{F}_M^{\leftarrow}\left(\Delta^{-1}\overline{F}_S(t)\right).$$

Again, by combining Proposition 2.4 (ii) and (iii) we have as $p \to 0$

$$\overline{F}_S^{\leftarrow}(p) \sim \overline{F}_M^{\leftarrow}\left(\Delta^{-1}\overline{F}_S \circ \overline{F}_S^{\leftarrow}(p)\right) \sim \overline{F}_M^{\leftarrow}(\Delta^{-1}p)$$

After rewriting the last equation in terms of the VaR function the result follows. Now we prove that the result is still valid when changing the VaR measure by the TVaR above, assuming the last exists. As we have that

$$\mathrm{VaR}_{1-u}(S) \sim \mathrm{VaR}_{1-\Delta^{-1}u}(M)$$

when $u \to 0$ then there exists a function $\epsilon(p)$, $\epsilon(p) \downarrow 0$ as $p \downarrow 0$ such that for any $0 < u \le p$

$$(1-\epsilon(p))\,\mathrm{VaR}_{1-\Delta^{-1}u}(M) \le \mathrm{VaR}_{1-u}(S) \le (1+\epsilon(p))\mathrm{VaR}_{1-\Delta^{-1}u}(M).$$

After integrating over u from 0 to p each side of the inequality above we get,

$$(1-\epsilon(p))\,\mathrm{TVaR}_{1-\Delta^{-1}p}(M) \le \mathrm{TVaR}_{1-p}(S) \le (1+\epsilon(p))\mathrm{TVaR}_{1-\Delta^{-1}p}(M),$$

and then as $\epsilon(p) \downarrow 0$ it is clear from the last inequality that

$$\mathrm{TVaR}_{1-p}(S) \sim \mathrm{TVaR}_{1-\Delta^{-1}p}(M)$$

when $p \to 0$. Remark that as we assume that marginal risks are almost surely positive we always have that

$$\{\max\{X_1, \ldots, X_d\} > t\} \subset \{X_1 + \cdots + X_d > t\} \subset \{\max\{X_1, \ldots, X_d\} > t/d\}$$

and thus $\delta(t) \leq \overline{F}_M(t/d)/\overline{F}_M(t)$. So that if $\overline{F}_M$ is regularly varying with index ρ then $\Delta \leq d^{-\rho}$.

□

Classes of random vectors such that the tail of the maximum is regularly or consistently varying will be given in Section 4 while in Section 5 we will provide a method to estimate Δ. The hypothesis of the convergence of $\delta(x)$ in Theorem 3.1 can be relaxed if we assume regularity of the maximum. Before explaining that point, we need to prove the following lemma.

Lemma 3.2. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a vector of positive r.v.s. If both $\overline{F}_M$ and $\overline{F}_S$ are regularly varying at infinity then they have the same variation index.*

Proof. Suppose that $\overline{F}_M \in \mathcal{RV}_\infty(-\rho_M)$ and that $\overline{F}_S \in \mathcal{RV}_\infty(-\rho_S)$, for some positive values ρ_S and ρ_M. As before, because $\mathbf{X}$ has positive components, $\overline{F}_M(t) \leq \overline{F}_S(t) \leq \overline{F}_M(t/d)$ for any $t > 0$ and then

$$1 \leq \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq \frac{\overline{F}_M(t/d)}{\overline{F}_M(t)}$$

for any $t > 0$. Regular variation of $\overline{F}_M$ implies then that

$$1 \leq \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq d^{\rho_M} + \epsilon \tag{3.1}$$

for some positive values ϵ, T and any $t > T$. Now let L_M and L_S be the slowly varying functions that satisfy

$$\overline{F}_M(t) = t^{-\rho_M} L_M(t) \quad \text{and} \quad \overline{F}_S(t) = t^{-\rho_S} L_S(t),$$

then

$$\frac{\overline{F}_S(t)}{\overline{F}_M(t)} = t^{\rho} \frac{L_S(t)}{L_M(t)},$$

with $\rho = \rho_M - \rho_S$. Rewriting last equation as

$$\frac{\overline{F}_S(t)}{\overline{F}_M(t)} = \frac{t^{\rho/2} L_S(t)}{t^{-\rho/2} L_M(t)},$$

allows us to check, by Proposition 1.3.6 in [9], that as $t \to \infty$ either $\frac{\overline{F}_S(t)}{\overline{F}_M(t)} \to \infty$ if $\rho > 0$ or $\frac{\overline{F}_S(t)}{\overline{F}_M(t)} \to 0$ if $\rho < 0$. As both possibilities contradict inequality (3.1) we must have $\rho = 0$ and thus $\rho_M = \rho_S$.

□

Theorem 3.3. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a vector of positive r.v.s. If the function $\delta(x) = \frac{\overline{F}_S(x)}{\overline{F}_M(x)}$ is continuous but not convergent and $\overline{F}_M$ and $\overline{F}_S$ are both regularly varying with negative indexes then*

$$\mathrm{VaR}_{1-\beta p_n}(S) \sim \mathrm{VaR}_{1-\Delta^{-1}\beta p_n}(M)$$

for some sequence $p_n \to 0$, some $\Delta \geq 1$ and any $\beta > 0$. The same result applies for the TVaR *if* $\mathrm{TVaR}_p(M)$ *exists for any p.*

Proof. We have $\overline{F}_M(t) \leq \overline{F}_S(t) \leq \overline{F}_M(t/d)$ for any $t > 0$, so $\overline{F}_M \in \mathcal{RV}_\infty(\rho)$, $\rho < 0$, implies that eventually $1 \leq \delta(t) \leq d^{-\rho}$. So if $\delta(t)$ is not convergent there exist two strictly increasing and unbounded real sequences $(u_n)_{n>0}$, $(v_n)_{n>0}$ such that $b(u_n) \to \Delta_1$ and $b(v_n) \to \Delta_2$ as $n \to \infty$, for some $1 \leq \Delta_1 < \Delta_2 \leq d^{-\rho}$. Without loss of generality we may suppose also that $\Delta(v_n) > \Delta$ and $\Delta(u_n) < \Delta$ for all $n > 0$ with $\Delta = (\Delta_1 + \Delta_2)/2$. As both sequences v_n and u_n are unbounded we can construct two natural sequences $(j_i)_{i>0}$ and $(k_i)_{i>0}$ such that $u_{k_i} < v_{j_i} < u_{k_{i+1}}$ for all $i = 0, 1, \ldots$. Continuity of $\delta(t)$ allows us then to pick up a strictly increasing and unbounded real sequence $(t_n)_{n>0}$, such that $u_{k_n} < t_n < v_{j_n}$ and such that $\Delta(t_n) = \Delta$. Then

$$\overline{F}_S(t_n) = \Delta \overline{F}_M(t_n),$$

for all $n > 0$. As both $\overline{F}_S$ and $\overline{F}_M$ are regularly varying with same index, for any $s > 0$

$$\overline{F}_S(st_n) \sim \Delta \overline{F}_M(st_n)$$

and then after applying Proposition A.2 (ii) we get

$$\overline{F}_S^{\leftarrow}(\beta p_n) \sim \Delta^{-\rho^{-1}} \overline{F}_M^{\leftarrow}(\beta p_n) \sim \overline{F}_M^{\leftarrow}(\Delta^{-1} \beta p_n),$$

where $p_n = 1/t_n$ and $\beta = 1/s$. Rewriting last equation with the VaR notation finishes the proof. The TVaR case follows exactly as in the proof of Theorem 3.1. □

In theory this theorem allows us to approximate the $\mathrm{VaR}_{1-p}(S)$ at any level p, by taking $\beta = p/p_n$, for n big. However, in practice, it will not be usable as the sequence p_n is in general not known.

4. On the regular and consistently variation of the tail of the max

In this section we explore several situations in which $\overline{F}_M$ is regularly or consistently varying. We also exhibit some classes of random vectors for which the limit Δ exists.

4.1. **Multivariate regular framework.**

Alink et al. ([3], [2] and [4]) studied the asymptotic behaviour of the tail of the sum when the marginals of the vector $\mathbf{X} = (X_1, \ldots, X_d)$ are identically distributed as one of the three extreme value families: Gumbel, Fréchet or Weibull and when the dependence within the vector is given by an Archimedean copula. Then Barbe et al. ([6]) generalized these results under the framework of the multivariate regular variation distributions. Their main contribution is the explicit calculation of the limit

$$\lim_{t \to \infty} \frac{\overline{F}_S(t)}{\overline{F}_1(t)},$$

where F_1 is the common distribution function of the marginal risks $X_1, \ldots, X_d$.

This kind of results suggest that we may approximate the VaR (and TVaR) of the sum simply by the VaR (and TVaR) of X_1. Our main results based on the maximum would then not be very interesting. This point will be detailed in Section 7.4 where it will be shown that maximum based estimation gives indeed better results than F_1 based one.

Let us recall the definition of multivariate regularly varying random vectors.

Definition 4 (Multivariate Regular Variation)**.** A random vector $\mathbf{X}$ is said to be multivariate regularly varying of index $\beta > 0$ if there exists a probability measure μ on $\Gamma_d = \left\{ \frac{\mathbf{x}}{\|\mathbf{x}\|} \, : \, \mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\} \right\}$, a function $b : (0, \infty) \to (0, \infty)$ and a scalar γ, such that for all $x > 0$ and all $A \subset \Gamma_d$,

$$\lim_{t\to\infty} t \Pr\left(\|\mathbf{X}\| > x b(t), \frac{\mathbf{X}}{\|\mathbf{X}\|} \in A \right) = \frac{\gamma}{x^\beta} \mu(A),$$

where $\gamma = \gamma(\|\cdot\|, b)$ depends on both the norm $\|\cdot\|$ used and the function b.

As shown in Barbe et al. (2006) ([6]), using the L^1 norm, $\|\mathbf{X}\|_1 = |X_1| + \cdots + |X_d|$, and $b(t) = F_1^{\leftarrow}(1 - 1/t)$ in Definition 4, one finds

$$\lim_{t\to\infty} \frac{\overline{F}_S(t)}{\overline{F}_1(t)} = \lim_{t\to\infty} \frac{\overline{F}_S(b(t))}{\overline{F}_1(b(t))} = \lim_{t\to\infty} t \Pr(\|\mathbf{X}\|_1 > b(t)) = \gamma(L^1, b).$$

Similarly, we can use the L^∞ norm, $\|\mathbf{X}\|_\infty = \max\{[X_1|, \ldots, |X_d|\}$ to get

$$\lim_{t\to\infty} \frac{\overline{F}_M(t)}{\overline{F}_1(t)} = \lim_{t\to\infty} t \Pr(\|\mathbf{X}\|_\infty > b(t)) = \gamma(L^\infty, b).$$

We conclude then that when $\mathbf{X}$ is multivariate regularly varying

$$\lim_{t\to\infty} \frac{\overline{F}_S(t)}{\overline{F}_M(t)} = \frac{\gamma(L^1, b)}{\gamma(L^\infty, b)} = \Delta.$$

So that, when $\mathbf{X}$ is multivariate regularly varying Theorem 3.1 applies.

We are also interested in random vectors whose coordinates are not identically distributed. Results for identically distributed marginals will not lead to results for arbitrary marginals. This is the purpose of the next section where different kinds of dependence structure are also considered.

4.2. **The case of independence or Archimedean copula dependence structure.**

In this section the results presented in appendix A are used to prove, in the case of independence and for random vectors with an Archimedean copula dependence structure, that the maximal survival function is regularly or consistently varying.

Let us remark that considering a random vector with regularly varying marginals does not imply that $\overline{F}_M$ is regularly varying. Nevertheless, some bounds may be obtained. Denote by $\overline{F}_i$ the survival function of X_i and assume that $\overline{F}_i \in \mathcal{RV}_\infty(-\rho_i)$ for $i = 1, \ldots, d$ with $\rho_i > 0$.

Fréchet bounds imply that for any $t > 0$

$$1 - \min_i \{F_i(t)\} \le \overline{F}_M(t) \le 1 - \left(1 - \sum_{i=1}^d \overline{F}_i(t) \right)_+ ,$$

where $(\cdot)_+$ stands for the positive part function, and so when t is such that $\overline{F}_i(t) \leq d^{-1}$ for all $i = 1, \dots, d$

$$\max_i\{\overline{F}_i(t)\} \leq \overline{F}_M(t) \leq \sum_{i=1}^d \overline{F}_i(t) \leq d \max_i\{\overline{F}_i(t)\}.$$

So

$$\frac{\max_i\{\overline{F}_i(tx)\}}{d\max_i\{\overline{F}_i(t)\}} \leq \frac{\overline{F}_M(tx)}{\overline{F}_M(x)} \leq \frac{d\max_i\{\overline{F}_i(tx)\}}{\max_i\{\overline{F}_i(t)\}}.$$

Then if there exists j such that eventually $\overline{F}_j(x) \geq \overline{F}_i(x)$ for any i and that for such a j, $\overline{F}_j \in \mathcal{RV}_\infty(-\rho_j)$, then for $\overline{F}_{M*}(x) = \liminf_{t\to\infty} \dfrac{\overline{F}_M(tx)}{\overline{F}_M(x)}$ and $\overline{F}^*_M(x) = \limsup_{t\to\infty} \dfrac{\overline{F}_M(tx)}{\overline{F}_M(x)}$,

$$d^{-1}x^{-\rho_j} \leq \overline{F}_{M*}(x) \leq \overline{F}^*_M(x) \leq dx^{-\rho_j}$$

$$\overline{F}_j(t) \leq \overline{F}_M(t) \leq \sum_{i=1}^d \overline{F}_i(t).$$

Now as (see e.g. [9]) $\sum_{i=1}^d \overline{F}_i(t)$ is in $\mathcal{RV}_\infty(-\rho_*)$ with $\rho_* = \min\{\rho_i : i = 1, \dots d\} = \rho_j$, then there exists g and h in $\mathcal{RV}_\infty(-\rho_j)$ such that

$$g(t) \leq \overline{F}_M(t) \leq h(t).$$

Under some additional hypothesis, we now get that $\overline{F}_M$ is regularly varying.

Proposition 4.1. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a vector of positive r.v.s. If $\overline{F}_j(x)/\overline{F}_1(x) \to 0$ when $x \to \infty$ for all $2 \leq j \leq d$, then $\overline{F}_M(x) \sim \overline{F}_1(x)$. Then if $\overline{F}_1 \in \mathcal{RV}_\infty(-\rho)$, $\rho > 0$ so does $\overline{F}_M$.*

Proof. It follows directly from the remark that

$$\overline{F}_M(t) \geq \overline{F}_1(t) \text{ and } \overline{F}_M(t) \leq \sum_{i=1}^d \overline{F}_i(t).$$

□

We now consider the case where the dependence structure between components of the random vector is given by an Archimedean copula and study conditions under which the tail of the maximum is regularly or consistently varying. We first recall the definition of Archimedean copula.

Definition 5. (Archimedean Copulas)
C is an Archimedean copula with generator ψ if

$$C(u_1, \dots, u_d) = \psi^{\leftarrow}\{\psi(u_1) + \cdots + \psi(u_d)\},$$

for all $u_i \in [0,1]$ and $i = 1, \dots, d$, where the generator ψ is a function $\psi : [0,1] \to [0,\infty]$ that satisfies:

(i) ψ is strictly decreasing with $\psi(1) = 0$;
(ii) The first k derivatives of $\psi^{\leftarrow}$ exist and are continuous;

(iii) For $k = 0, 1, \dots, d$,

$$(-1)^k \frac{d^k \psi^{\leftarrow}(t)}{dt^k} \geq 0, \text{ for all } t > 0.$$

Proposition 4.2. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a random vector with regularly varying tails, $\overline{F}_i \in \mathcal{RV}_\infty(-\rho_i)$, $i = 1, \dots, d$. Suppose that the copula C of $\boldsymbol{X}$ is Archimedean with generator ψ and $\psi \in \mathcal{RV}_1(-\theta)$ with $\theta > 0$.*

Then the tail of the maximum is regularly varying, $\overline{F}_M \in \mathcal{RV}_\infty(-\rho)$, with index $\rho = \min_i\{\rho_i\}$.

Proof. By definition

$$\overline{F}_M(x) = 1 - \psi^{-1}\left(\sum_{i=1}^{d} \psi(F_i(x))\right).$$

Lemma A.4 gives

$$\psi \circ F_i \in \mathcal{RV}_\infty(-\rho_i\theta)$$

for each $i = 1, \dots d$, and then

$$\phi :\equiv \sum_{i=1}^{d} (\psi \circ F_i) \in \mathcal{RV}_\infty(-\rho\theta)$$

with $\rho = \min_i\{\rho_i\}$. Lemma A.3 (ii) implies that

$$1 - \psi^{-1} \in \mathcal{RV}_0(\theta^{-1})$$

and applying again Lemma A.4 gives

$$\overline{F}_M \equiv (1 - \psi^{-1}) \circ \phi \in \mathcal{RV}_\infty(-\rho)$$

as required.

□

Corollary 4.3. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a vector of positive r.v.s. If the marginals are independent and their survival functions $\overline{F}_i \in \mathcal{RV}_\infty(-\rho_i)$, $\rho_i > 0$, $i = 1, \dots, d$, then*

$$\overline{F}_M \in \mathcal{RV}_\infty(-\rho)$$

with $\rho = \min_i\{\rho_i\}$.

Proof. It suffices to notice that the independent copula is an Archimedean copula with generator $\psi(t) = -\ln(t)$. □

Proposition 4.4. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a random vector with consistently varying marginal tails. Suppose that the copula C of $\boldsymbol{X}$ is Archimedean with generator ψ and that $x \mapsto \psi(1 - 1/x)$ is consistently varying. Then the tail of the maximum is consistently varying.*

Proof. The proof is exactly the same as before, but instead of Lemmas A.3 and A.4, we use here Propositions 2.4 (i) and 2.1 (i). The closure by sums of the consistently varying functions stated in Proposition 2.1 (ii) is also required. □

In the case of survival Archimedean copulas, we derive the regular variation property for the tail of the maximum M and the minimum m in the following proposition.

Proposition 4.5. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a random vector with regularly varying marginal tails, $\overline{F}_i \in \mathcal{RV}_\infty(-\rho_i)$, $i = 1, \ldots, d$. Suppose that the copula C of $\boldsymbol{X}$ is the survival copula of an Archimedean copula $\tilde{\mathcal{C}}$ with generator ψ with $\psi \in \mathcal{RV}_0(\theta)$, $\theta > 0$. Then*

(i) *The tail of the minimum of $\boldsymbol{X}$, $m := \min\{X_1, \ldots, X_d\}$, is regularly varying, $\overline{F}_m \in \mathcal{RV}_\infty(-\rho^*)$, with index $\rho^* = \max_i\{\rho_i\}$.*

(ii) *If there exists a distribution F such that*

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}(x)} = a_i, \tag{4.1}$$

where $0 \le a_i \le 1$ for all $i = 1, \ldots d$ and at least one is non-zero, then the tail of the maximum is regularly varying, $\overline{F}_M \in \mathcal{RV}_\infty(-\rho)$, with index $\rho = \min_i\{\rho_i\}$.

Proof. (i) Notice first that

$$\overline{F}_m(x) = \Pr(X_1 > x, \ldots, X_d > x).$$

The survival copula dependence of $\mathbf{X}$ implies

$$\overline{F}_m(x) = \tilde{C}(\overline{F}_1(x), \ldots, \overline{F}_d(x)),$$

and then by definition,

$$\overline{F}_m(x) = \psi^{-1}\left(\sum_{i=1}^{d} \psi(\overline{F}_i(x))\right).$$

Lemma A.4 gives, for each $i = 1, \ldots, d$, $\psi \circ \overline{F}_i \in \mathcal{RV}_\infty(\rho_i \times \theta)$ and thus

$$\sum_{i=1}^{d} \psi(\overline{F}_i(x)) \in \mathcal{RV}_\infty(\rho^* \times \theta)$$

with $\rho^* = \max\{\rho_i, i = 1, \ldots d\}$. Combining Lemmas A.3 (i) and A.4 as in the proof of Proposition 4.2 allows us to conclude that $\overline{F}_m \in \mathcal{RV}_\infty(-\rho^*)$.

(ii) For $i = 1, \ldots, d$ set $A_i = \{X_i \le x\}$, and $A = \bigcap_{i=1}^d A_i$. Then, by the inclusion-exclusion principle, we have that

$$\overline{F}_M(x) = \Pr(A^c) = \sum_{k=1}^{d} (-1)^{k+1} \sum_{i_1,\ldots,i_k \in \{1,\ldots,d\}} \Pr(A^c_{i_1}, \ldots, A^c_{i_k}). \tag{4.2}$$

Notice that by (i), for each $k = 1, \ldots, d$ and each set $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$ each addend in the sum above, in absolute value, is regularly varying. That is because if $m_I = \min\{X_{i_1}, \ldots, X_{i_k}\}$ then

$$\overline{F}_{m_I}(x) = \Pr(A^c_{i_1}, \ldots, A^c_{i_k}) \in \mathcal{RV}_\infty(-\rho^*)$$

with $\rho^* = \max\{\rho_{i_1}, \ldots, \rho_{i_k}\}$. However as not all elements in the sum are positive we can just conclude that $\overline{F}_M$ is the difference of two regularly varying functions, and even if the difference is known to be positive we cannot conclude directly on its regularity (see for example [20]). Nevertheless if there exists a distribution function F such that

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}(x)} = a_i$$

whence $0 \leq a_i \leq 1$ for all $i = 1, \ldots d$ with at least one a_i non-zero, then the regularity of $\overline{F}_M$ can be proved. In this case for $k = 1, \ldots, d$ and each set $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, d\}$ we have

$$\sum_{i \in I} \psi(\overline{F}_i(x)) \sim \sum_{i \in I} \psi(a_i \overline{F}(x)) \sim \psi(\overline{F}(x)) \sum_{i \in I} a_i^{-\theta}$$

and by Lemma A.3 and Proposition 2.4 (iii)

$$\overline{F}_{m_I}(x) = \psi^{\leftarrow}\left(\sum_{i \in I} \psi(\overline{F}_i(x))\right) \sim \left(\sum_{i \in I} a_i^{-\theta}\right)^{-1/\theta} \overline{F}(x). \tag{4.3}$$

Then combining equations (4.2) and (4.3) gives

$$\lim_{x \to \infty} \frac{\overline{F}_M(x)}{\overline{F}(x)} = \sum_{k=1}^{d} (-1)^{k+1} \sum_{i_1, \ldots, i_k \in \{1, \ldots, d\}} \left(\sum_{j=1}^{k} a_{i_j}^{-\theta}\right)^{-1/\theta}.$$

Remark that for any index i and any $x \in \mathbb{R}$, $\overline{F}_M(x) \geq \overline{F}_i(x)$, so that

$$\frac{\overline{F}_M(x)}{\overline{F}(x)} \geq \frac{\overline{F}_i(x)}{\overline{F}(x)} \overset{x \to \infty}{\longrightarrow} a_i.$$

Since we assume that at least for one index i, $a_i > 0$, we conclude that $\overline{F}_M \in \mathcal{RV}_\infty(-\rho)$ where $-\rho$ is the index of regularity of $\overline{F}$. Let us check that the index of $\overline{F}$ should be $\rho = \min_i\{\rho_i\}$. By hypothesis there exists i such that $a_i \neq 0$, and thus (4.1) implies $\overline{F} \in \mathcal{RV}_\infty(-\rho_i)$. If there exists $j \neq i$ such that $\rho_j < \rho_i$ then $\overline{F}_j(x)/\overline{F}(x) \to \infty$ as $x \to \infty$ which contradicts that $0 \leq a_j \leq 1$. This shows that $\rho = \min_i\{\rho_i\}$. □

4.3. **On the convergence of $\delta(x) = \overline{F}_S(x)/\overline{F}_M(x)$.** We complete this section by some remarks on the convergence of $\delta(x) = \overline{F}_S(x)/\overline{F}_M(x)$. As above (see proof of Lemma 3.2), we have

$$\overline{F}_M(t) \leq \overline{F}_S(t) \leq \overline{F}_M(t/d)$$

so that

$$1 \leq \liminf_{t \to \infty} \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq \limsup_{t \to \infty} \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq \limsup_{t \to \infty} \frac{\overline{F}_M(t/d)}{\overline{F}_M(t)}.$$

When $\overline{F}_M$ is regularly varying with index $-\rho < 0$ we have that the quotient $\overline{F}_S(t)/\overline{F}_M(t)$ is eventually bounded, i.e., for any $\epsilon > 0$, there exists T such that

$$1 \leq \frac{\overline{F}_S(t)}{\overline{F}_M(t)} \leq d^\rho + \epsilon$$

for any $t > T$. Then, in this case, the monotonicity of $F_S(x)/F_M(x)$ would be sufficient to prove the convergence of $\delta(x)$.

Now, if we assume also that $\overline{F}_S$ is regularly varying, by Lemma 3.2 we must have that the index of variation is the same as in $\overline{F}_M$ and then it can be easily shown that $L(t) = \frac{\overline{F}_S(t)}{\overline{F}_M(t)}$ is a slowly varying function. However this does not allow us to conclude anything on the convergence because a slowly varying function, even bounded, does not converge necessarily (see for example [27]).

In the simple case where one of the components of $\mathbf{X}$ has a strictly heavier tail than the others then the convergence holds.

Proposition 4.6. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a vector of positive regularly varying r.v.s with indexes $-\rho_i < 0$, $i = 1, \ldots, d$. If $\rho_1 < \rho_i$, for any $i = 2, \ldots, d$ then*

$$\frac{\overline{F}_S(x)}{\overline{F}_M(x)} \to 1,$$

when $x \to \infty$.

Proof. By Proposition 4.1 it follows that $\overline{F}_M(x) \sim \overline{F}_1(x)$ as $x \to \infty$. Similarly it is easy to show that $\overline{F}_S(x) \sim \overline{F}_1(x)$ and thus the proposition follows. □

5. Approximation of the limit Δ

In this section, we show how to estimate the limit Δ using samples of $\mathbf{X}$. We will use this estimation to approximate $\mathrm{VaR}_{1-p}(S)$, for different values of p close to 0 using Theorem 3.1.

Recall that δ is the real valued function defined by $\delta(t) = \overline{F}_S(t)/\overline{F}_M(t)$ and continue to denote by Δ its limit at infinity if it exists.

If a sample of $\mathbf{X}$ is available, the function δ can be estimated using the empirical cumulative distribution function (e.c.d.f.) of S and M. As we assume that F_M can be easily calculated by the c.d.f. $\mathbf{F}$ of the portfolio, at least two versions of the empirical delta should interest us:

$$\widehat{\delta}(t) = \frac{1 - \widehat{F}_S(t)}{1 - F_M(t)} \qquad \text{and} \qquad \tilde{\delta}(t) = \frac{1 - \widehat{F}_S(t)}{1 - \widehat{F}_M(t)}$$

where $\widehat{F}_S$ and $\widehat{F}_M$ are the e.c.d.f.s of S and M respectively, based on the sample of $\mathbf{X}$. The first version $\widehat{\delta}$ may be more tractable statistically, while the second $\tilde{\delta}$ has the nice property that $\tilde{\delta} \geq 1$. In order to obtain some insight on the convergence of δ to its limit Δ, we plot, in Figure 1, functions $\widehat{\delta}$ and $\tilde{\delta}$ for four different models.

In the first model we notice that the limit $\delta(t)$ seems to be one but the convergence is not fast enough to consider using this limit to approximate $VaR_p(S)$ even for higher confidence levels p. For the second model the convergence is a lot faster, $\delta(t)$ seems to be close to its limit for t greater than the VaR at the 95% confidence level. The two models in the lower side behave the same as the ones in the upper side.

The models on the right side of Figure 1 correspond to cases where our method will be applicable: the limit Δ is reached by $\widehat{\delta}(t)$ for t near the $VaR_{0.95}$. These models exhibit a strong dependence combined with at least one of the marginal risks with a very heavy tail. Even if this is a limitation of our method we should remark that this kind of models are also those where Monte Carlo methods are less efficient to approximate the VaR or the TVaR, so that it may be interesting to have an alternative method of approximation.

On a possible estimator of Δ

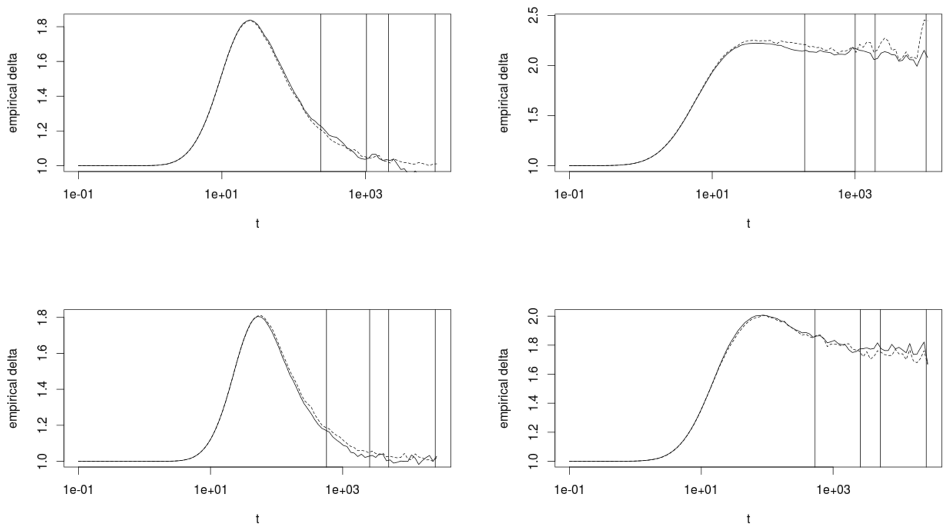

FIGURE 1. Four plots of $\widehat{\delta}$ (solid) and $\tilde{\delta}$ (dashed) for different models, based on samples with size 10^4. Vertical lines are displayed at the empirical VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%. Each model is a sum of 10 Pareto distributions with different tail indexes and different dependence structures. From top-left to bottom-right, we find: 1) independent Pareto distributions with tail index one; 2) the tail index is still one but dependence is given by a Gumbel copula of parameter 1.5; 3) independent Pareto distributions: five with tail index one and the other five with tail index 3; 4) the same as 3) but dependence is given by a Gumbel copula of parameter 1.5.

Let $(S_1, \dots, S_n)$ be an i.i.d. sample of S. According to Donsker's Theorem, the empirical process

$$\sqrt{n}(\widehat{F}_S(t) - F_S(t))$$

converges in distribution to a Gaussian process with zero mean and covariance given by

$$F_S(t_1) - F_S(t_1)F_S(t_2)$$

for $t_1 \leq t_2$. Thus, given any sequence $0 < t_1 < \cdots < t_k$, the vector

$$\sqrt{n}\left(\widehat{\delta}(t_1) - \delta(t_1), \dots, \widehat{\delta}(t_k) - \delta(t_k)\right)$$

converges in law to a centred Gaussian vector with covariances given by

$$\frac{F_S(t_i) - F_S(t_i)F_S(t_j)}{(1 - F_M(t_i))(1 - F_M(t_j))} = \frac{\delta(t_j)}{1 - F_M(t_i)} - \delta(t_i)\delta(t_j)$$

for any $i \leq j$. As a consequence

$$\sqrt{n}\left(\frac{1}{k}\sum_{i=1}^{k}\widehat{\delta}(t_i) - \frac{1}{k}\sum_{i=1}^{k}\delta(t_i)\right)$$

converges to a normal distribution with zero mean and variance

$$\frac{1}{k^2}\sum_{1\le i\le j\le k}\left\{\frac{\delta(t_j)}{1-F_M(t_i)}-\delta(t_i)\delta(t_j)\right\}. \tag{5.1}$$

If we assume that the values t_i are large enough, the approximation $\delta(t_i)\approx\Delta$ holds for each $i=1,\ldots,k$ and the variance (5.1) can be approximated by

$$\frac{\Delta}{k^2}\sum_{i=1}^{k}\left\{\frac{i}{1-F_M(t_i)}\right\}-\frac{\Delta^2(k+1)}{2k}.$$

In practice we should plot points $\left(S_{(i)},\widehat{\delta}(S_{(i)})\right)$ where $S_{(1)}<\cdots<S_{(n)}$ is the ordered sample of S and then choose a threshold in such a way that the approximation $\delta(S_{(n-i)})\approx\Delta$ holds for any $0\le i\le k$. The choice of the threshold is a recurrent and difficult problem in EVT, for which few theoretical results exist and are generally hardly applicable in practice. We propose then to estimate Δ by

$$\widehat{\Delta}=\frac{1}{k}\sum_{i=1}^{k}\widehat{\delta}(S_{(n-i)}). \tag{5.2}$$

As an example, the behavior of $\widehat{\delta}(x)$ for the Pareto-Clayton model, which will be described in Section 6, may be seen on Figure 2. The estimation $\widehat{\Delta}$ is represented by the solid line while dashed lines are for the estimated 95% confidence interval. See also Figure 3 for the shape of the δ function and the limit Δ.

6. Some Explicit Calculations

In this section we will consider a simplified model in order to obtain explicit formulas for F_S and F_M and so better understand the scope and the limitations of our Δ estimation. The model is described by the following compound process: let Λ be a positive random variable and let $\mathbf{X}=(X_1,\ldots,X_d)$ be a random vector such that

$$\Pr(X_1>x_1,\ldots,X_d>x_d\,|\,\Lambda=\lambda)=\prod_{i=1}^{d}e^{-\lambda x_i}, \tag{6.1}$$

for each $x_1,\ldots,x_d\ge 0$.

That means that conditionally on the value of Λ the marginals of $\mathbf{X}$ are independent and exponentially distributed. The final distribution of $\mathbf{X}$ will not have, in general, independent marginals and they will not be exponential either. Actually the dependence structure of $\mathbf{X}$ and its marginal distributions will depend on the distribution of Λ.

Some particular Λ distributions define some well-known models in which the explicit calculation of F_S and F_M is possible. For example when Λ is Gamma distributed, then the marginals of $\mathbf{X}$ are of Pareto type with dependence given by a survival Clayton copula. When Λ is Levy distributed the marginals will be Weibull distributed with a Gumbel survival copula. These models have been studied in [24, 30] and used in [1] where explicit formulas for ruin probabilities have been derived. In [10, 22], explicit results

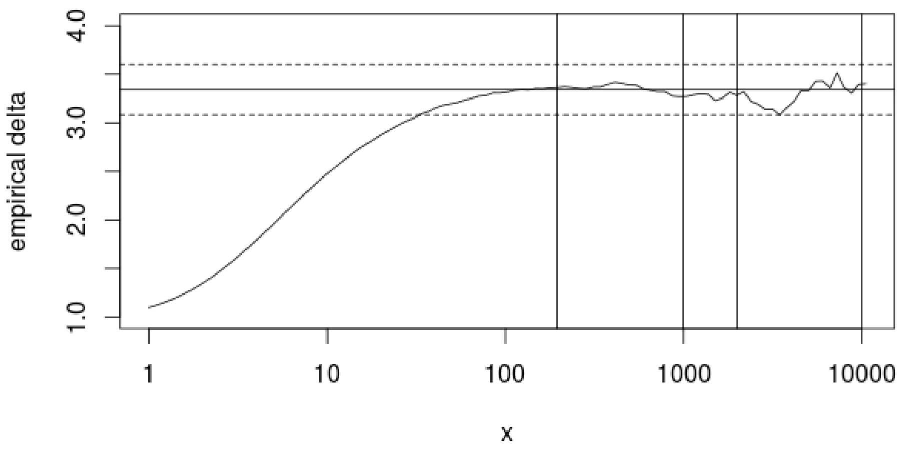

FIGURE 2. Shape of the $\widehat{\delta}$ function of the Pareto Clayton model with parameters $\alpha = 1, \beta = 1$ and $d = 10$ based on samples of size 10^4. Vertical lines are displayed at the empirical VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%. The estimation $\widehat{\Delta}$ with its estimated 95% confidence interval is represented by the horizontal lines.

for the minimum of some risk indicators are obtained for this kind of models. We also would like to mention that the computation of the VaR for this model is given in [13].

Let us consider the case where Λ is Gamma(α, β) distributed with density

$$f_\Lambda(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

In this case, the X_i's are Pareto(α, β) distributed with tail given by

$$\overline{F}_i(x) = \left(1 + \frac{x}{\beta}\right)^{-\alpha}$$

and the dependence structure is described by a survival Clayton copula with parameter $1/\alpha$. Through this paper we will refer to this model as a Pareto Clayton vector with parameters (α, β). This model is a particular Multivariate Pareto of type II with location parameters $\mu_i = 0$ and scale parameters $\sigma_i = \beta$ for $i = 1, \ldots, d$ (see [30]).

In the Pareto Clayton model, the exact distribution function of $S = \sum_{i=1}^d X_i$ can be calculated. Conditionally to $\Lambda = \lambda$, $\sum_{i=1}^d X_i$ is Gamma$(1/\lambda, d)$ distributed, distribution also known as the Erlang distribution. Then, as here we are assuming that Λ is Gamma(α, β) distributed, the total distribution of S is the result of compounding two Gamma distributions, more precisely

$$S \sim \text{ Gamma}(1/\Lambda, d) \text{ where } \Lambda \sim \text{ Gamma}(\alpha, \beta).$$

It is well known that the result of this compound distribution is the so-called Beta prime distribution (see [14]). The c.d.f. of S can be expressed in terms of F_β, the c.d.f. of the Beta$(d\beta, \alpha)$ distribution, as

$$F_S(x) = F_\beta\left(\frac{x}{1+x}\right).$$

Naturally, the inverse of F_S can also be expressed in function of the inverse of the Beta distribution

$$F_S^{\leftarrow}(p) = \frac{F_\beta^{\leftarrow}(p)}{1 - F_\beta^{\leftarrow}(p)}.$$

In this example, the δ function is explicitly calculated (see Figure 3). Moreover, computer algebra softwares allow us to calculate explicitly the limit Δ for specified parameters.

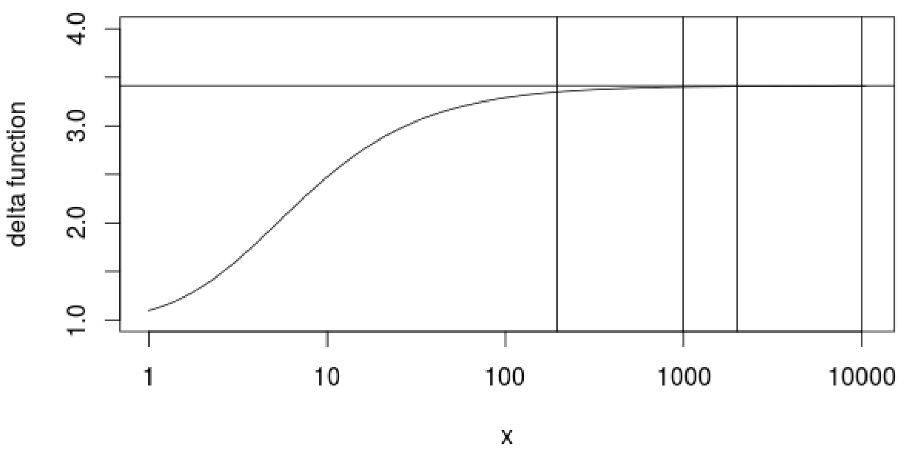

FIGURE 3. Shape of the δ function of the Pareto Clayton model, with parameters $\alpha = 1, \beta = 1$ and $d = 10$. Vertical lines are displayed at the VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%. The limit $\Delta \approx 3.4142$ is represented by the horizontal line.

In order to see how fast the function δ converges to its limit Δ, we plot the function $p \mapsto \delta(\mathrm{VaR}_p(S))$ for different values of the parameter α and different dimensions d (see Figure 4). We remark that $\delta(x)$ is already very close to Δ when $x = \mathrm{VaR}_{0.95}(S)$, for $\alpha \leq 2.5$. The lower the value α, the flatter the tail of δ and thus the limit Δ is attained rapidly. Remark that the lower the levels of α, the heavier the tails of the Pareto marginals. Finally, this plot confirms the intuition that for heavier marginals the tail of the sum is better approximated by the tail of the maximum.

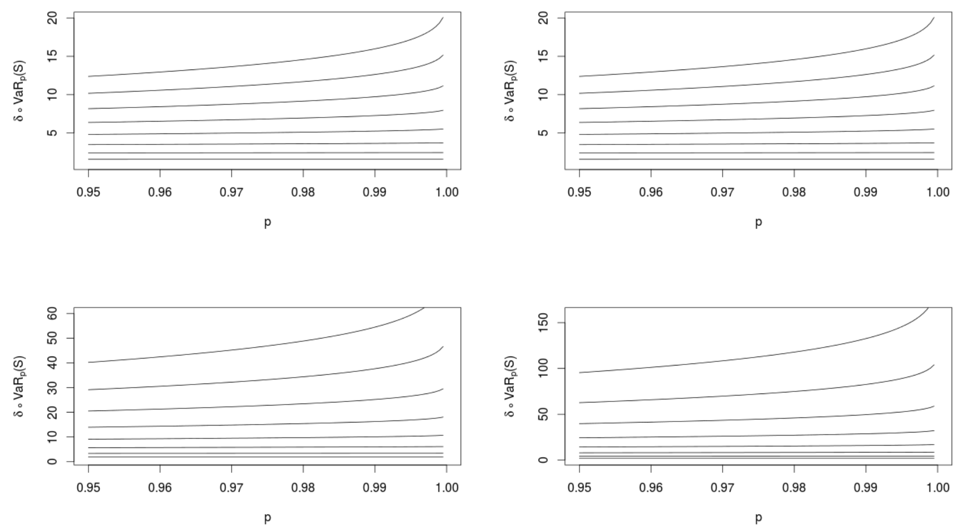

FIGURE 4. Four plots of the $p \mapsto \delta(\mathrm{VaR}_p(S))$ function of the Pareto Clayton model for dimensions $d = 2, 6, 10$ and 14 (from top-left to bottom-right) are represented. For each dimension, the curves with $\alpha = 0.5, 1, 1.5, 2, 2.5, 3, 3.5$ and 4 are plotted and they can be seen from bottom to top on each chart.

7. Some numerical examples

In this section we show how the ideas presented in the above section can help to estimate in practice the VaR and the TVaR of a sum at confidence levels close to 1. We compare our results with the empirical quantiles of the sample used to estimate Δ and with three different quantile estimation methods from the Extreme Value Theory.

We first consider the Pareto Clayton model presented in Section 6 (dimension 2 and 10), where exact values for the Value-at-Risk are computable. Then we test our method with a different model where exact values are not known. We compare the estimation done via the Δ-limit estimation with other common quantile estimation methods:

(1) The direct Monte Carlo quantile estimation (MC).
(2) The quantile estimation from a GPD fitted distribution where parameters are estimated using maximum likelihood method (GPD 1).
(3) The quantile estimation from a GPD fitted distribution where parameters are estimated using the moment method (GPD 2).
(4) The high quantile estimate based on a method by Weissman [29] (Weiss.).

In order to study the performance of our estimator and to compare with the main competitors, we consider the root-mean-squared error (RMSE) loss function. When n estimations have been performed, it is defined by

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(\widehat{\text{VaR}}_p(S^i) - \text{VaR}_p(S)\right)^2},$$

where $\widehat{\text{VaR}}_p(S^i)$ represents the estimate of $\text{VaR}_p(S)$ for any of the different methods presented above, on the ith sample.

7.1. **Pareto Clayton model dimension 2.** Here we consider the model presented in Section 6. We first consider $d = 2$ and $\alpha = 1$ which corresponds to a model with Pareto marginals with $\alpha = 1$ and dependence given by a survival Clayton copula with parameter $\theta = 1$.

In Table 1, the exact VaR at different confidence levels (from 95% to 99.95%) is presented. In Table 2 and Table 3, we present the RMSE criterion in percentage of the real value based on 1000 simulations at different confidence levels. At each simulation a sample of size 10^4 in Table 2 and size 10^5 in Table 3 is used to estimate the VaR. On each method (New, GPD 1, GPD 2 and Weiss) the threshold used on each estimation corresponds to the empirical 95% quantile. Clearly, in term of RMSE, our method performs better than classical methods at each confidence level, even for very high levels. When increasing the size of the sample (10^5 instead of 10^4) classical methods improve but our method still produces the best results.

VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
194.5	994.5	1994.5	9994.5	19994.5

TABLE 1. Exact Value-at-Risk at different confidence levels on the Pareto Clayton model in dimension $d = 2$ with $\alpha = 1$.

Method	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
New	1.9%	1.7%	1.7%	1.7%	1.7%
MC	4.4%	10.3%	14.1%	38.2%	76.2%
GPD 1	11.3%	8.5%	11.8%	23.8%	30.2%
GPD 2	4.4%	11.1%	15.1%	25.1%	29.9%
Weiss.	4.4%	11.2%	15.1%	25.0%	29.6%

TABLE 2. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size 10^4 is used to estimate the VaR.

7.2. **Pareto Clayton model dimension 10.** We consider again the Pareto-Clayton model but here $d = 10$ and $\alpha = 1$ which corresponds to a model with Pareto marginals with $\alpha = 1$ and dependence given by a survival Clayton copula with parameter $\theta = 1$. Results are presented in Tables 4, 5 and 6. As above, on each method (New, GPD 1, GPD 2 and Weiss) the threshold used on each estimation corresponds to the empirical 95% quantile. We

Method	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
New	0.7%	0.5%	0.6%	0.6%	0.6%
MC	1.4%	3.1%	4.4%	9.7%	14.4%
GPD 1	5.2%	2.6%	3.6%	7.2%	8.9%
GPD 2	1.4%	3.7%	4.7%	7.7%	9.1%
Weiss.	1.4%	3.9%	4.9%	7.7%	9.0%

Table 3. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size 10^5 is used to estimate the VaR.

mention that even in dimension 10, the estimation remains efficient for high level quantiles.

VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
194.5	994.5	1994.5	9994.5	19994.5

Table 4. Exact Value-at-Risk at different confidence levels on the Pareto Clayton model in dimension $d = 10$ with $\alpha = 1$.

Method	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
New Method	8.4%	7.8%	7.7%	7.7%	7.7%
MC	4.5%	10.1%	14.5%	43.6%	85.5%
GPD 1	10.7%	8.5%	12.1%	25.0%	32.1%
GPD 2	4.5%	11.3%	15.6%	26.5%	31.8%
Weiss.	4.5%	11.4%	15.5%	26.1%	31.2%

Table 5. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size 10^4 is used to estimate the VaR.

Method	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
New	2.6%	2.2%	2.2%	2.3%	2.3%
MC	1.4%	3.2%	4.6%	10.1%	14.8%
GPD 1	4.3%	2.7%	3.8%	7.4%	9.2%
GPD 2	1.4%	3.6%	4.8%	7.8%	9.2%
Weiss.	1.4%	4.1%	5.2%	7.9%	9.1%

Table 6. RMSE in percentage of the real value based on 1000 simulations. At each simulation a sample of size 10^5 is used to estimate the VaR.

We also remark that our method is more efficient than classical ones from level 0.99.

7.3. **A model with 150 different Pareto marginals and Gumbel copula.** We apply now our method to a model where the exact value of $\mathrm{VaR}_p(S)$ is not known. The model is composed of 150 marginals Pareto(α_i, β_i) distributed with parameters $\alpha_i = (3 - i \mod (3))/2$ and $\beta_i = 5 - i \mod (5)$ for $i = 1, \ldots, 150$, where $i \mod (j)$ denote the reminder of i divided by j. The model is then composed of fifty Pareto marginals of tail index 0.5, fifty of tail index 1 and fifty with tail index 1.5, and different scale parameters within $1, 2, \ldots, 5$. The dependence structure is given by a Gumbel copula of parameter 1.5.

Table 7 presents the VaR estimation based on a classical Monte Carlo quantile estimation with a sample of size 3×10^8. We assume this estimation is the "real VaR" in the computation of the RMSE presented in Table 8. On each method (New, GPD 1, GPD 2 and Weiss) the threshold used on each estimation corresponds to the empirical 99% quantile. It is notable that our method is very stable with respect to others and is more efficient to approximate the VaR_p from $p = 0.99$.

VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
8.1981e06	3.2770e07	8.1545e08	3.2561e09

TABLE 7. Estimated Value-at-Risk at different confidence levels for the model described in Section 7.3 estimated with a sample of size 3×10^8.

Method	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
New	5.0%	4.9%	5.0%	5.0%
MC	6.2%	9.2%	21.2%	30.9%
GPD 1	5.9%	7.7%	12.4%	16.3%
GPD 2	5.9%	7.9%	13.1%	15.4%
Weiss.	5.9%	7.9%	13.0%	15.3%

TABLE 8. RMSE in percentage of the estimated VaR presented in Table 7 based on 1000 simulations. At each simulation a sample of size 10^5 is used to estimate the VaR.

7.4. **Comparison of the method using** $\max(X)$ **vs** X_1**.** The method of estimation of the Value-at-Risk of the sum proposed in this paper relies on the convergence of the function $\delta(t) = \overline{F}_S(t)/\overline{F}_M(t)$. When the convergence is assured and it is fast enough, it has been shown that the proposed method gives accurate and stable estimations of the VaR at high levels. In theory, similar results could be obtained if the maximum M is replaced by X_1 where X_1 is assumed to have the heaviest tail in the vector X. In this section we compare numerically the estimation of the VaR using, on one side, $\delta(t) = \overline{F}_S(t)/\overline{F}_M(t)$ and, on the other side, $\delta'(t) = \overline{F}_S(t)/\overline{F}_{X_1}(t)$, i.e we compare the approximation of $\mathrm{VaR}_{1-p}(S)$ by $\mathrm{VaR}_{1-p/\Delta}(M)$ and

$\mathrm{VaR}_{1-p/\Delta'}(X_1)$ where Δ and Δ' are the approximated limits of $\delta(t)$ and $\delta'(t)$ respectively estimated using (5.2).

We first consider the model $(X_1, \ldots, X_{10})$ where X_1 is Pareto distributed with $\alpha = 0.9$ and $X_2, \ldots, X_{10}$ are Pareto distributed with $\alpha = 1$. The dependence structure is given by a Gumbel copula with parameter 2. Empirical δ and δ' functions are displayed in Figure 5.

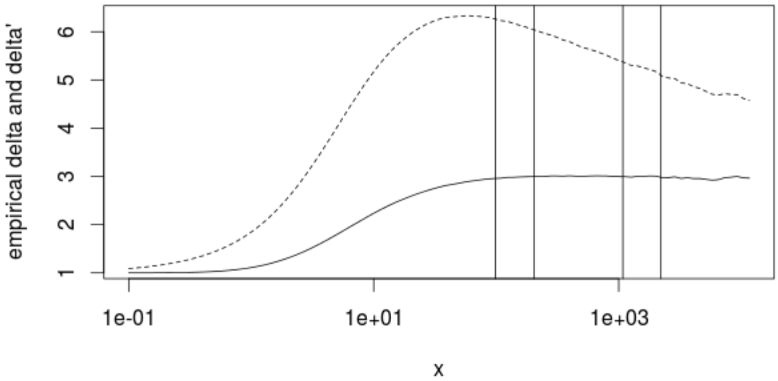

Figure 5. Shape of an empirical $\delta(x)$ (solid) and $\delta'(x)$ (dashed) functions based on 10^5 simulations. Vertical lines are displayed at the empirical VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%.

The δ function becomes almost horizontal before the VaR of the sum at the 95% confidence level whereas δ' does not seem to be close to the limit on the displayed range. Then, the estimation of the VaR using δ' seems to be not accurate. This is confirmed by Table 9 where some VaR estimations are presented. From now on, the threshold used for the Δ and the Δ' approximations using formula (5.2) corresponds to the 95% empirical quantile and for each estimation a sample of size 10^5 is generated.

	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
MC (3×10^8)	200	1058	2166	11201	22809
New method using $\max(X)$	203 (2%)	1067 (2%)	2188 (2%)	11665 (5%)	24083 (6%)
New method using X_1	188 (6%)	1126 (7%)	2432 (12%)	14549 (30%)	31428 (38%)

Table 9. First line: Monte Carlo VaR estimation using 3×10^8 simulations. Second and third lines: mean and RMSE of 1000 VaR estimations using the max and the Δ' approximations. The RMSE is presented in % of the MC estimation.

Even in the case where all the marginal risks are equal the use of the max seems to give better results. We consider the model $(X_1, \ldots, X_{10})$ where all the X_i's are Pareto distributed with the same index $\alpha = 1$. The dependence structure is given by a Gumbel copula with parameter 2. Empirical δ and δ' functions are displayed in Figure 6.

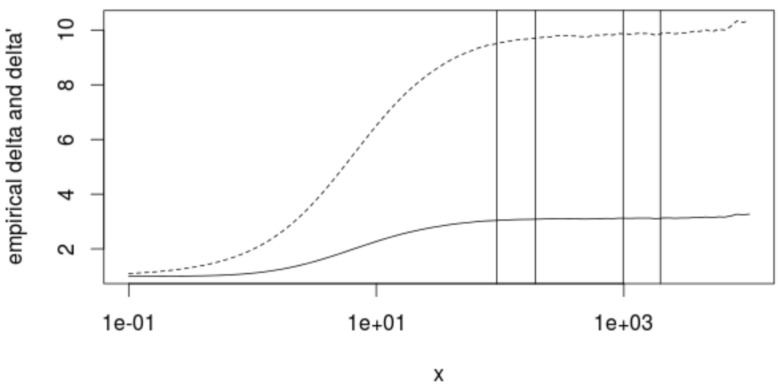

FIGURE 6. Shape of an empirical $\delta(x)$ (solid) and $\delta'(x)$ (dashed) functions based in 10^5 simulations. Vertical lines are displayed at the empirical VaR of the sum at confidence levels 95%, 99%, 99.5%, 99.9%

As above the δ function seems to converge faster than δ' but in this case the difference is not as important as in Figure 5. In table 9 some VaR estimations are presented. Again, estimations provided by using the estimation of Δ are of better quality than the ones provided by using the estimation of Δ'.

	VaR 95%	VaR 99%	VaR 99.5%	VaR 99.9%	VaR 99.95%
MC (3×10^8)	196	1003	1996	9977	19931
New method using $\max(X)$	202 (4%)	1068 (7%)	2189 (10%)	11671 (17%)	24097 (21%)
New method using X_1	188 (5%)	1126 (13%)	2434 (22%)	14556 (46%)	31444 (58%)

TABLE 10. First line: Monte Carlo VaR estimation using 3×10^8 simulations. Second and third lines: mean and RMSE of 1000 VaR estimations using the max and the Δ' approximations. The RMSE is presented in % of the MC estimation.

8. Conclusion

In this paper, we give some conditions under which the tail distribution of the sum can be approximated by using the tail of the maximum of a vector. We show how, basically whenever the maximum is consistently varying, the VaR or the TVaR on high levels for the sum can be approximated, by first estimating a limiting constant Δ. The models in which our results can be applied include those where marginals are consistently varying and such that dependence is given by an Archimedean copula or survival copula. We do not require the marginals to be identically distributed and the method works for very high dimensions d ($d = 150$ for exemple). Our method gives a good approximation for the VaR and the TVaR when the convergence of $\delta(x)$ to Δ is fast enough. This generally happens when at least one of the marginal risks is strongly heavy tailed and when the dependence is strong. In particular, the method is not suitable e.g. for the case of two independent Pareto distributions. We also remark that the models for which our method applies correspond generally to those where Monte Carlo approximations are less efficient ant there so is a real need for alternative methods.

Appendix A. Some general properties on regularly varying functions and their inverses

The results given in this section are used in Section 4.2.

Proposition A.1. (i) *Suppose f is non-decreasing, and $f \in \mathcal{RV}_\infty(\rho)$, $\rho > 0$. Then*

$$f^{\leftarrow} \in \mathcal{RV}_\infty(\rho^{-1}) \quad \textit{and} \quad f \circ f^{\leftarrow}(x) \sim f^{\leftarrow} \circ f(x) \sim x.$$

(ii) *Suppose f is non-increasing, and $f \in \mathcal{RV}_\infty(\rho)$, $\rho < 0$. Then*

$$f^{\leftarrow} \in \mathcal{RV}_0(-\rho^{-1}), \quad f \circ f^{\leftarrow}(1/x) \sim 1/x \quad \textit{and} \quad f^{\leftarrow} \circ f(x) \sim x.$$

Proof. (i): For the proof see [26] Chapter 2. (ii): Now suppose f is non-increasing and that $f \in \mathcal{RV}_\infty(\rho)$, $\rho < 0$. Then $(1/f) \in \mathcal{RV}_\infty(-\rho)$, and by the non-decreasing case $(1/f)^{\leftarrow} \in \mathcal{RV}_\infty(-1/\rho)$ which is equivalent to say that $f^{\leftarrow}(1/x) \in \mathcal{RV}_\infty(-1/\rho)$ and this by definition implies $f^{\leftarrow} \in \mathcal{RV}_0(-1/\rho)$. To show that $f \circ f^{\leftarrow}(1/x) \sim 1/x$ simply note that this is equivalent to show that $(1/f) \circ (1/f)^{\leftarrow}(x) \sim x$.

□

Proposition A.2. (i) *Suppose f_1 and f_2 are non-decreasing, $\rho-$varying at infinity, $\rho > 0$. Then for $0 < c < \infty$*

$$f_1(x) \sim c f_2(x) \quad \textit{iff} \quad f_1^{\leftarrow}(x) \sim c^{-\rho^{-1}} f_2^{\leftarrow}(x).$$

(ii) *Suppose f_1 and f_2 are non-increasing, ρ-varying at infinity, $\rho < 0$. Then for $0 < c < \infty$*

$$f_1(x) \sim c f_2(x) \quad \textit{iff} \quad f_1^{\leftarrow}(1/x) \sim c^{-\rho^{-1}} f_2^{\leftarrow}(1/x). \tag{A.1}$$

Proof. The proof for the non-decreasing functions can be found on [26] Chapter 2. The non-increasing case then follows immediately by noticing that $f_1 \sim cf_2$ iff $(1/f_1) \sim (1/c)(1/f_2)$ iff $(1/f_1)^{\leftarrow} \sim (1/c)^{\rho^{-1}}(1/f_2)^{\leftarrow}$ iff $f_1^{\leftarrow}(1/x) \sim c^{-\rho^{-1}} f_2^{\leftarrow}(1/x)$.

□

Lemma A.3. *Let $\psi : [0,1] \to \mathbb{R}_+$ be a non-increasing function with $\psi(0) = +\infty$ and $\psi(1) = 0$.*

- *(i) If $\psi \in \mathcal{RV}_0(\theta_0)$ for $\theta_0 > 0$ then*
$$\psi^{\leftarrow} \in \mathcal{RV}_\infty(-\theta_0^{-1})$$
- *(ii) If $\psi \in \mathcal{RV}_1(-\theta_1)$ for $\theta_1 > 0$ then*
$$(1 - \psi^{\leftarrow}) \in \mathcal{RV}_0(-\theta_1^{-1})$$

Proof. i) Set $f(x) = \psi(1/x)$, then $f \in \mathcal{RV}_\infty(\theta_0)$. By Proposition A.1 (i) $f^{\leftarrow} \in \mathcal{RV}_\infty(\theta_0^{-1})$, and as $f^{\leftarrow} = 1/\psi^{\leftarrow}$ then $\psi^{\leftarrow} \in \mathcal{RV}_\infty(-\theta_0^{-1})$.
ii) Set $f(x) = \psi(1 - 1/x)$, then $f \in \mathcal{RV}_\infty(-\theta_1)$. By Proposition A.1 (ii) $f^{\leftarrow} \in \mathcal{RV}_0(\theta_1^{-1})$ and as $f^{\leftarrow} = 1/(1 - \psi^{\leftarrow})$ then $(1 - \psi^{\leftarrow}) \in \mathcal{RV}_0(-\theta_1^{-1})$

□

Lemma A.4. *If $f \in \mathcal{RV}_\infty(-\rho)$ with $f(t) \downarrow 0$ as $t \to \infty$, $\psi_0 \in \mathcal{RV}_0(\theta_0)$ and $\psi_1 \in \mathcal{RV}_1(-\theta_1)$ then*
$$\psi_0 \circ f \in \mathcal{RV}_\infty(-\rho \times \theta_0) \quad \textit{and} \quad \psi_1 \circ (1 - f) \in \mathcal{RV}_\infty(-\rho \times \theta_1).$$

Proof. Notice that as $f(tx) \sim t^{-\rho} f(x)$ then
$$\psi \circ f(tx) \sim \psi(t^{-\rho} f(x)) \sim t^{-\rho\times\theta_0} \psi \circ f(x).$$
Notice that the first $\sim$ is consequence from the Proposition 2.4 (iii). To prove the other equation let $h(x) = \psi_1(1-1/x)$, then by definition $h \in \mathcal{RV}_\infty(-\theta_1)$. The proof follows by noticing that $\psi_1 \circ (1 - f) = h \circ (1/f)$. □

References

[1] Hansjörg Albrecher, Corina Constantinescu, and Stéphane Loisel. Explicit ruin formulas for models with dependence among risks. *Insurance: Mathematics and Economics*, 48:265–270, 2011.

[2] Stan Alink, Matthias Löwe, and Mario V. Wüthrich. Diversification of aggregate dependent risks. *Insurance: Mathematics and Economics*, 35(1):77–95, 2004.

[3] Stan Alink, Matthias Löwe, and Mario V. Wüthrich. Analysis of the expected shortfall of aggregate dependent risks. *ASTIN Bulletin*, 35(1):25–43, 2005.

[4] Stan Alink, Matthias Löwe, and Mario V. Wüthrich. Diversification for general copula dependence. *Statistica Neerlandica*, 61(4):446–465, November 2007.

[5] P. Arbenz, P. Embrechts, and G. Puccetti. The AEP algorithm for the fast computation of the distribution of the sum of dependent random variables. *Bernoulli*, 17(2):562–591, 2011.

[6] Philippe Barbe, Anne-Laure Fougères, and Christian Genest. On the Tail Behavior of Sums of Dependent Risks. *ASTIN Bulletin*, 36(2):361–373, 2006.

[7] Carole Bernard, Ludger Rüschendorf, and Steven Vanduffel. Value-at-Risk bounds with variance constraints. 2013.

[8] Carole Bernard and Steven Vanduffel. A new approach to assessing model risk in high dimensions. *Journal of Banking and Finance*, 2015.

[9] Nicholas H Bingham, Charles M Goldie, and Jef L Teugels. *Regular variation*, volume 27. Cambridge university press, 1989.

[10] Peggy Cénac, Stephane Loisel, Véronique Maume-Deschamps, and Clémentine Prieur. Risk indicators with several lines of business: comparison, asymptotic behavior and applications to optimal reserve allocation. *Annales de l'ISUP*, 58(3), 2014.

[11] Daren BH Cline and Gennady Samorodnitsky. Subexponentiality of the product of independent random variables. *Stochastic Processes and their Applications*, 49(1):75–98, 1994.
[12] Hélène Cossette, Marie-Pier Côté, Mélina Mailhot, and Etienne Marceau. A note on the computation of sharp numerical bounds for the distribution of the sum, product or ratio of dependent risks. *Journal of Multivariate Analysis*, 130:1–20, 2014.
[13] Michel Dacorogna, Leila El Bahtouri, and Marie Kratz. Explicit diversification benefit for dependent risks. preprint, 2014.
[14] Satya D Dubey. Compound gamma, beta and F distributions. *Metrika*, 16(1):27–31, 1970.
[15] Paul Embrechts, Claudia Klüppelberg, and Thomas Mikosch. *Modelling extremal events: for insurance and finance*, volume 33. Springer, 1997.
[16] Paul Embrechts, Johanna Neslehová, and Mario V Wüthrich. Additivity properties for value-at-risk under archimedean dependence and heavy-tailedness. *Insurance: Mathematics and Economics*, 44(2):164–169, 2009.
[17] Paul Embrechts, Giovanni Puccetti, and Ludger Rüschendorf. Model uncertainty and VaR aggregation. *Journal of Banking & Finance*, 37(8):2750–2764, 2013.
[18] Anne-Laure Fougères and Cécile Mercadier. Risk measures and multivariate extensions of Breiman's theorem. *Journal of Applied Probability*, 49:364–384, 2012.
[19] Jaap Geluk and Qihe Tang. Asymptotic tail probabilities of sums of dependent subexponential random variables. *Journal of Theoretical Probability*, 22(4):871–882, 2009.
[20] Slobodanka Janković and Tatjana Ostrogorski. Differences of slowly varying functions. *Journal of mathematical analysis and Applications*, 247(2):478–488, 2000.
[21] Dominik Kortschak and Hansjörg Albrecher. Asymptotic results for the sum of dependent non-identically distributed random variables. *Methodology and Computing in Applied Probability*, 11(3):279–306, 2009.
[22] Véronique Maume-Deschamps, Didier Rullière, and Khalil Saïd. On the optimal capital allocation by minimizing multivariate risk indicators. 2015.
[23] Quang Huy Nguyen and Christian Y Robert. New efficient estimators in rare event simulation with heavy tails. *Journal of Computational and Applied Mathematics*, 261:39–47, 2014.
[24] David Oakes. Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, 84(406):487–493, 1989.
[25] Sidney I Resnick. *A probability path*. Birkhauser Boston, Cambridge, MA, 1998.
[26] Sidney I. Resnick. *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling*. Number v. 10 in Heavy-tail Phenomena: Probabilistic and Statistical Modeling. Springer, 2007.
[27] Slavko Simić. A class of C^∞ slowly varying functions. *Publikacija Elektrotehnickog fakulteta - serija: matematika*, (11):59–70, 2000.
[28] Stefan Weinzierl. Introduction to monte carlo methods. *arXiv preprint hep-ph/0006269*, 2000.
[29] Ishay Weissman. Estimation of parameters and large quantiles based on the k largest observations. *Journal of the American Statistical Association*, 73(364):812–815, 1978.
[30] Hsiaw-Chan Yeh. The frailty and the Archimedean structure of the general multivariate Pareto distributions. *Bulletin Institute of Mathematics Acedemia Sinica*, 2(3):713–729, 2007.

Université de Lyon, Université Lyon 1, Laboratoire SAF EA 2429, SCOR SE
E-mail address: acuberos@scor.com

Université de Lyon, Université Lyon 1, Institut Camille Jordan ICJ UMR 5208 CNRS
E-mail address: esterina.masiello@univ-lyon1.fr

Université de Lyon, Université Lyon 1, Institut Camille Jordan ICJ UMR 5208 CNRS
E-mail address: veronique.maume@univ-lyon1.fr